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Pinning control of clustered complex networks with different size

Chenbo Fu, Jinbao Wang, Yun Xiang, Zhefu Wu, Li Yu, Qi Xuan *

College of Information Engineering, Zhejiang University of Technology, Hangzhou, China

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ABSTRACT

In pinning control of complex networks, it is found that, with the same pinning effort, the network can be better controlled by pinning the large-degree nodes. But in the clustered complex networks, this preferential pinning (PP) strategy is losing its effectiveness. In this paper, we demonstrate that in the clustered complex networks, especially when the clusters have different size, the random pinning (RP) strategy performs much better than the PP strategy. Then, we propose a new pinning strategy based on cluster degree. It is revealed that the new cluster pinning strategy behaves better than RP strategy when there are only a smaller number of pinning nodes. The mechanism is studied by using eigenvalue and eigenvector analysis, and the simulations of coupled chaotic oscillators are given to verify the theoretical results. These findings could be beneficial for the design of control schemes in some practical systems.

1. Introduction

Many complex system can be represented by networks, and controlling the complex network is one of the most challenging problems in science and engineering fields. Moreover, the need of regulating the behavior of complex system with interacting units is a common feature of many natural and artificial systems. Over the past decade, great effort has been devoted to understanding the relationship between the network topology and dynamic process, and also designing the control strategies on them [1–3]. However, due to the complexity of network structure and the chaotic feature of node dynamics, the systems always become quite sensitive, making it difficult to be controlled. Hitherto, many control methods have been proposed to control chaotic systems [4–9], one of which is the pinning control that is now widely adopted in controlling chaotic networks [3,6,7,10–20]. For a network with a given coupling scheme, pinning control means to control the network state to a goal state or a specific targeting orbit by pinning only part of the system variables [6,7].

In order to tame a complex network through pinning control method, the control strategy should be carefully selected or specially designed. Usually, it is assumed that pinning the large-degree nodes can get a better control result [21–24], but this traditional pinning strategy may lose its efficiency when the network has a specific topology, e.g., modular structure. Many realistic networks have such structure, e.g., the network of books on American politics [25,26], Zachary's karate club network [27], and dolphin social network [28], here we call them clustered networks for simplicity. Traditional pinning control strategy may fail to deal with these networks, in certain situations, even random pinning strategy behaves better than it. Recently, Miao et al. [13] have studied the pinning controllability of clustered networks through Master Stability Function (MSF) and found that when the number of pinning nodes increases gradually, the ascending pinning scheme behaves more

* Corresponding author.
E-mail address: xuanqi@zjut.edu.cn (Q. Xuan).

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effectively than descending pinning scheme, which is similar to the results in the networks without modular structure [2]. However, in their study, all the clusters have similar size, while in many real networks, the size of cluster may be varied. For instance, the network about books on American politics can be roughly divided into three clusters, i.e., two relatively large clusters represent liberal and conservative, respectively, and one small cluster represents centrist or unaligned [26]. Such heterogeneity on cluster size requires us to propose better pinning control strategy.

In the present work, motivated by the above discussion, we will investigate the pinning control in clustered network with the cluster size highly varied. For the pinning control of a network, there are two basic questions: *Which nodes should be controlled? And how many nodes should be chosen?* These two questions will also be carefully investigated in this paper. In particular, we will study different pinning strategies in clustered networks, and by the method of eigenvalue and eigenvector analysis, we will show how the pinning location influences the pinning controllability. Our main finding is that, in the clustered networks with the cluster size highly varied, the traditional preferential pinning (PP) strategy behaves even worse than the random pinning (RP) strategy. Thus, we propose a new pinning strategy, namely cluster pinning (CP) strategy, which generally behaves much better than the PP and RP strategies in this kind of networks. Interestingly, we find that the pinning location can be easily observed in the eigenvector space and these eigenvector components can help to inference the critical eigenvalue which represents the controllability of the system. These findings have the potential to extend our knowledge of pinning control in the clustered networks and can be of utility in diverse domains such as disease control [29] and air traffic control [30].

The rest of our paper are organized as follows: in Section 2 we give our model of clustered network pinning control and study the phenomena by applying control strategies in real and artificial network. In Section 3, we study the mechanism of pinning strategy by using the method of eigenvector analysis. In Section 4, we present our new pinning strategy and compare it with other pinning strategies. Finally in Section 5, we provide our discussion and conclusion.

2. Method and observation

We consider the following model with the network of pinning control:

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + \varepsilon \sum_{j=1}^{N} a_{i,j}[\mathbf{H}(x_i) - \mathbf{H}(x_j)] + \eta \sum_{m\in V} \delta_{im}[\mathbf{H}(\mathbf{x}_T) - \mathbf{H}(\mathbf{x}_i)] \tag{1}$$

where $i, j = 1, 2, \ldots, N$ are node indices, $\mathbf{F}$ is the function of node dynamics and $\mathbf{H}$ is the coupling function, ε and η represent the uniform coupling strength and the pinning strength, respectively. The network structure is captured by the adjacency matrix $\mathbf{A}$, with $a_{i,j} = 1$ if nodes i and j are directly connected and $a_{i,j} = 0$ otherwise. Here, the network is supposed to be undirected and unweighted, therefore the degree of node i can be read as $k_i = \sum_{j=1}^{N} a_{i,j}$. $\mathbf{x}_T$ is the target orbit, that is, the assumed controlled state of the whole network. Denoting f as the ratio of the pinning nodes to all nodes of the network, thus the number of pinning nodes is $N_p = f \times N$, then the set of pinning nodes is denoted by $V = \{n_i\}$, with $i = 1, \ldots, N_p$ as the indices of these pinning nodes. Specifically, if node i is pinned in the network, we have $\delta_{im} = 1$ in Eq. (1), otherwise, $\delta_{im} = 0$. Here, for simplicity, the dynamic of controller is set the same as the network nodes, i.e. $\dot{\mathbf{x}}_T = \mathbf{F}(\mathbf{x}_T)$.

Treating the controller as an additional node of the network, the pinning problem can be considered as a network synchronization problem and the original system can be embedded into an extended network with $(N + 1)$ nodes [22]. Under such consideration, the enlarged network can be characterized by the following matrix:

$$B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} & b_{1(N+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{iN} & b_{i(N+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} & b_{N(N+1)} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This enlarged network can be regarded as a special weighted and directed network. In this network, the controller has the index $N + 1$, with the value $b_{i(N+1)} = \eta/\varepsilon$ in the last column if $i \in V$ and $b_{i(N+1)} = 0$ otherwise. Generally, the linear stability of the target synchronization state $\{\mathbf{x}_i(t) = \mathbf{x}_T,\ i = 1, 2, \ldots, N\}$ can be analyzed by MSF [31–33], which shows the synchronizability of a network is generally determined by the extreme eigenvalues of the network coupling matrix. Then, Eq. (1) can be converted and diagonalized into an enlarged dynamic system with $N + 1$ decoupled blocks:

$$\dot{\mathbf{y}}_i = [\mathbf{DF}(\mathbf{x}_T) - \varepsilon\lambda_i \mathbf{DH}(\mathbf{x}_T)]\mathbf{y}_i, \tag{2}$$

where $\mathbf{y}_i,\ i = 1, 2, \ldots, N + 1$ represent the modes that are transverse to the synchronous manifold $\mathbf{x}_T$, and $\mathbf{DF}(\mathbf{x}_T)$ and $\mathbf{DH}(\mathbf{x}_T)$ are the Jacobian matrices of the corresponding vector functions evaluated on $\mathbf{x}_T$. $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_{N+1}$ are the eigenvalues of the coupling matrix $\mathbf{C} = \mathbf{B} - \mathbf{DI}$, $\mathbf{D} = (d_1, d_2, \ldots, d_{N+1})^T$, where $d_i = \sum_{j=1}^{N+1} b_{ij}$, is the coupling intensity of node i and $\mathbf{I}$ is the identity matrix of dimension $N + 1$. The coupling matrix $\mathbf{C}$ is Laplacian matrix and according to Ref. [31], this matrix satisfies $\sum_{j=1}^{N+1} C_{ij} = 0$, which is referred to as a diffusive condition. The mode associated with

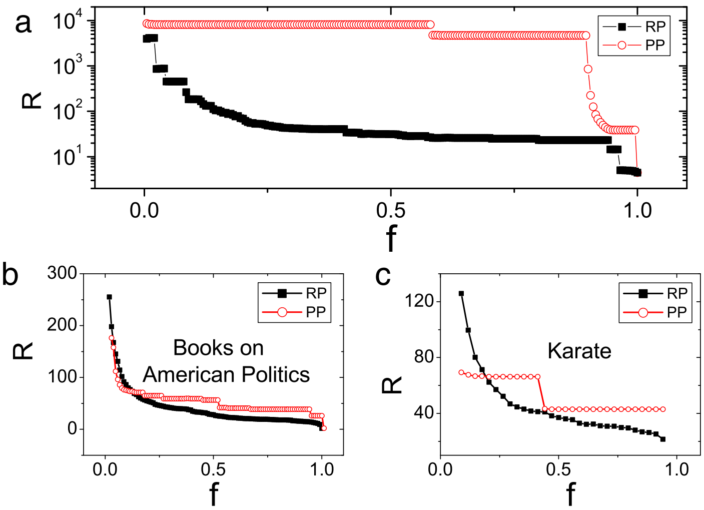

Fig. 1. (Color online) The phenomena of pinning control in clustered complex network, with coupling strength $\varepsilon = 1$ and pinning strength $\eta = 25$. The controllability of the network by adopting the RP strategy (black square) and the PP strategy (red circle) in (a) an artificial network with 1000 nodes and three clusters of size $N_1 = 100$, $N_2 = 300$, and $N_3 = 600$ respectively; (b) the network of books on American politics and (c) Zachary's karate club network. Each data is averaged over 64 realizations.

$\lambda = 0$ represents the motion parallel to the synchronous manifold and in order to keep the network synchronizable, it is required that all other eigenmodes $\mathbf{y}_i$, $i = 2, \ldots, N+1$ are damping with time, i.e., the Lyapunov exponent of each eigenmode is negative, $\Lambda(\sigma \equiv -\varepsilon\lambda_i) < 0$. Previous studies of MSF have shown that, for the typical nonlinear oscillators, the value of Λ is only negative within a bounded region in parameter space, $\sigma \in (\sigma_1, \sigma_2)$ [33]. In particular, we should have $\varepsilon\lambda_2 < \sigma_2$ and $\varepsilon\lambda_{N+1} > \sigma_1$ simultaneously, which leads to the following necessary condition for network synchronization: $R = \lambda_{N+1}/\lambda_2 < \sigma_1/\sigma_2 = R_c$. The smaller of the eigenratio R, the more likely of the enlarged network to be synchronized, and the more efficiently for the pinning control. Thus, the task of pinning optimization is reduced to the decrease of R by choosing the pinning strategies. The aim of this work is to explore an effective pinning strategy on clustered networks.

Having specified the pinning model and defined the controllability measurement, now based on numerical simulations, we start to compare two pinning strategies on clustered network, the traditional preferential pinning (PP) strategy and random pinning (RP) strategy. To simulate, we first generate a network of $N = 1000$ nodes, which consists of three clusters with size $N_1 = 100$, $N_2 = 300$, $N_3 = 600$. Each cluster is generated by the ER random network model [34], and in this situation, the average degrees for the three clusters are $\langle k_1 \rangle \approx 10$, $\langle k_2 \rangle \approx 30$ and $\langle k_3 \rangle \approx 30$, respectively. Meanwhile, to make the cluster feature clear, nodes inside each cluster are connected by a larger probability, $p_{in} = 0.1$, while nodes from different clusters are connected by a much smaller probability, $p_{out} = 0.01$. In our simulation, the PP and RP strategies are operated as follows. For the PP strategy, we first sort the nodes by a descending order of their degrees, then the nodes are pinned according to their descending order, i.e., the N_p nodes of largest degree are chosen as pinning nodes. For the RP strategy, nodes are pinned in a random fashion, the N_p nodes are randomly chosen as pinning nodes, following a uniform distribution. The controllability of RP and PP strategies are plotted in Fig. 1(a), where it is shown that, by increasing the pinning ratio f, the RP strategy can get a better controllability, while the PP strategy seems to have a stage-like controllability. This phenomenon may be attributed to the modular structure of the network and can also be found in the real clustered networks, e.g., the network of books on American politics [25,26] and Zachary's karate club network [27], as shown in Fig. 1(b) and (c), respectively. The book network consists of $N = 105$ nodes and can be roughly divided into three communities. And the club network consists of $N = 34$ nodes and can be roughly divided into two communities. It should be noted that, when the pinning ratio is extremely small, e.g., the pinning nodes are fewer than the clusters, the PP strategy behaves better than the RP strategy. It is reasonable because, in this case, no matter which nodes are pinned, they cannot explore all the clusters in network and thus the pinning results will be similar on the networks without modular structure, i.e., pinning the nodes of largest degree will be the most efficient way.

3. Mechanism analysis

The fact that the RP strategy behaves better than the PP strategy when the number of pinned nodes is relatively large, e.g., larger than the number of clusters, can be attributed to the locations of the pinned nodes. When the cluster size varies, the larger cluster will probably have more pinned nodes for the PP strategy than the RP strategy. For instance, suppose in a clustered network, one cluster is much larger than the others, then most pinning nodes for the PP strategy will fall into the largest cluster; while for the RP strategy, some pinning nodes may also fall into other clusters. Thus, to answer the question *which nodes should be controlled* in a qualitative manner, we generate a network simply consists of two clusters, one has size

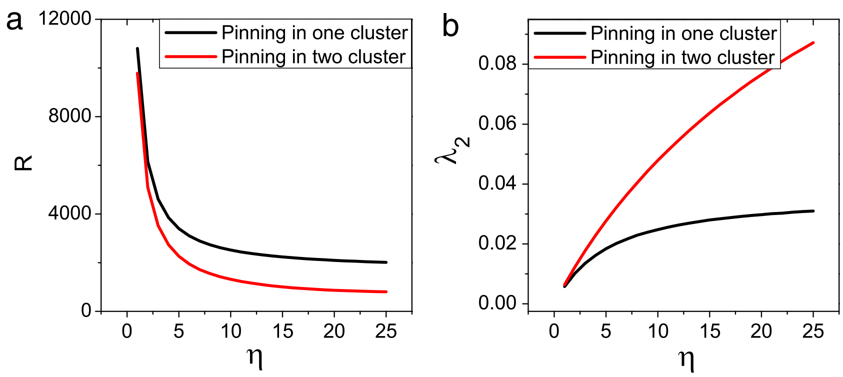

Fig. 2. (Color online) For a network consisting of two clusters with size $n_1 = 100$ and $n_2 = 200$, (a) the network eigenratio and (b) the smallest nonzero eigenvalue λ_2 as a function of pinning strength η.

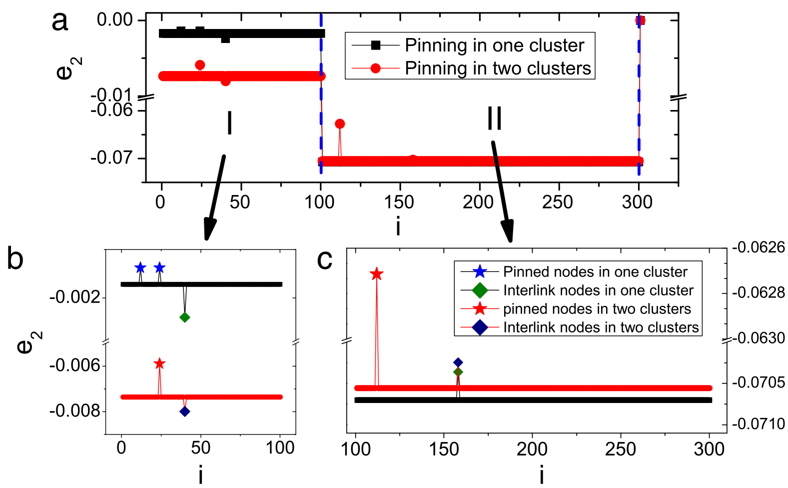

Fig. 3. (Color online) The eigenvector of the clustered network, the parameters are set the same as Fig. 2. (b) and (c) are zoom-in plots of the two clusters in the network, respectively.

$N_1 = 100$ and the other $N_2 = 200$. For simplification, we just consider two pinning nodes in the network, i.e., $N_p = 2$. In the first case, we put both of them in one cluster, e.g., in cluster one; in second case, we put one pinning node in cluster one and the other in cluster two. While in both cases, the pinning nodes are randomly selected. As we discussed in Section 2, for the general case of bounded MSF function, the pinning controllability is jointly determined by the ratio of the smallest nonzero eigenvalue λ_2 to the largest eigenvalue λ_{N+1} of the coupling matrix of the enlarged network. Regarding this, it is necessary to analyze the relationship between the pinning controllability and the location of the pinned nodes.

The role of location is presented in Fig. 2(a) and (b). It can be clearly seen that, with the same pinning effect, the different locations of pinned nodes will lead to quite different eigenvalues. Specifically, as the pinning strength increasing, the eigenratio in the second case (nodes are pinned in two clusters) is always smaller than that in the first case (nodes are pinned in just one cluster), as shown in Fig. 2(a), which means that the system is much easier to be controlled in the second case. Moreover, we also plot the smallest nonzero eigenvalue λ_2 as a function of pinning strength. Here, we just exhibit the variation of λ_2 because the variation of λ_{N+1} is very small, thus the variation of eigenratio is mostly depend on λ_2. As shown in Fig. 2(b), the smallest nonzero eigenvalue λ_2 in the second case grows more quickly than that in the first case, as the pinning strength increasing, which results in the reversed behave in eigenratio and well agrees with the results in Fig. 2(a). This indicates that here λ_2 plays most important role in the controllability of the network.

As far as the network eigenvectors are concerned, a significant change caused by the modular structure is that the elements of the second eigenvector are of stage-like distribution [35–37]. The eigenvalue spectra of C and its transpose C^T are identical. Let $\mathbf{e}_2$ $(e_1, e_2 \ldots, e_{1L}, \ldots, e_{1p}, e_{N_1}, e_{N_1+1}, \ldots, e_{2L}, \ldots, e_{2p}, \ldots, e_N, e_{N+1})$ be the normalized eigenvector associated with λ_2 of C^T. Where e_i means the ith element of eigenvector $\mathbf{e}_2$, N_1 means the size of cluster one, e_{iL} is associated with the nodes linked to the other cluster and e_{ip} is associated with the pinning nodes. Since $C^T\mathbf{e}_2 = \lambda_2\mathbf{e}_2$, we have $\lambda_2 = \mathbf{e}_2^T C^T \mathbf{e}_2 = \sum_{i,j=1}^{N+1} e_i C_{i,j} e_j$, and we can rewrite this by $(C - \lambda_2\mathbf{I})\mathbf{e}_2 = 0$. For a clustered network, the elements in $\mathbf{e}_2$ have a special distribution: $e_i \approx e_{1n}$ if node i is located in cluster one, and $e_j \approx e_{2n}$ if node j is located in cluster two, as demonstrated in Fig. 3(a). In order to facilitate the analysis, here we suppose that the clusters are fully connected. In the first case, for those normal nodes in cluster two, i.e., the nodes that are not pinned or linked to the other cluster, corresponding to the normal lows in $(C - \lambda_2 \mathbf{I})\mathbf{e}_2$, we have:

$$(N_2 - 1 - \lambda_2)e_{2n} - (N_2 - 1 - 1)e_{2n} - e_{2L} = 0, \tag{3}$$

which can be further simplified as

$$\lambda_2 = 1 - \frac{e_{2L}}{e_{2n}}. \tag{4}$$

For the second case, we have

$$(N_2 - 1 - \lambda_2')e_{2n}' - (N_2 - 1 - 1 - 1)e_{2n}' - e_{2p} - e_{2L} = 0, \tag{5}$$

which can be simplified as

$$\lambda_2' = 1 - \frac{e_{2p}}{e_{2n}'} + 1 - \frac{e_{2L}'}{e_{2n}'}. \tag{6}$$

It should be noted that there are many ways to define the λ_2 and λ_2', e.g., λ_2 can also be defined with e_{1n}, e_{1p} and e_{1L}. Here, we write the formulas with the second cluster eigenvector elements due to the following reasons. First, it is easy to compare. From Fig. 2, it can be seen that except those special nodes (pinning nodes and interlinked nodes) in cluster two, the value of other eigenvector elements are very close, $|e_{2n}'| - |e_{2n}| \approx 1.5 \times 10^{-4}$. This makes us easy to compare the variation of the smallest nonzero eigenvalue caused by pinning location. Second, it is easy to calculate. In the first form of eigenvector elements of cluster two, as presented in Eq. (4), the formula just contains two variables, but in the other forms, it will contain three or more variables. It should be noted that here we focus on the network with distinct modular structure, for these networks, the connections inside clusters are much denser than those between clusters, which makes the clusters more like independent subnetworks. Thus, when the number of pinning nodes is quite small, the eigenvector of the cluster with pinning nodes is similar to that of the cluster with no pinning nodes, e.g., e_{2n}' in the second case and e_{2n} in the first case (in our example, $e_{2n}' \approx -0.07055$ and $e_{2n} \approx -0.0707$). Furthermore, in our simulation, only the locations of pinning nodes are different, the locations of interlinked nodes are the same, so we get $e_{2L}'/e_{2n}' \sim e_{2L}/e_{2n}$. As a result, from the Eqs. (4) and (6), we have

$$\Delta\lambda = \lambda_2' - \lambda_2 = 1 - \frac{e_{2p}}{e_{2n}'} - \frac{e_{2L}'}{e_{2n}'} + \frac{e_{2L}}{e_{2n}} \approx 1 - \frac{e_{2p}}{e_{2n}'} > 0, \tag{7}$$

meaning the smallest nonzero eigenvalue of the second case is always larger than that of the first case, which well agrees with the Fig. 2(b). This can be confirmed by our simulation, where we get $\lambda_2' = 0.116$, $\lambda_2 = 0.005$, and $e_{2p} = -0.063$, $e_{2n'} = -0.071$, thus $\Delta\lambda = \lambda_2' - \lambda_2 = 0.111 \approx e_{2p}/e_{2n}'$.

4. New pinning strategy and numerical simulation

The RP strategy behaves better than the PP strategy mainly because the pinning nodes selected by the RP strategy may fall into more clusters than those selected by the PP strategy. But the pinning nodes in each cluster may not located in the best places. For this reason, we propose a new pinning strategy which based on the modular structure of the network and we call it cluster pinning (CP) strategy. Usually, for a network without modular structure, it is best to choose the nodes of largest degree to control [21–24]. Thus, two main steps of CP strategy are as follows:

- First, we divided the pinning number N_p into groups according to the cluster size:

$$N_{pi} = (N_i/N) \times N_p. \tag{8}$$

- Then in each cluster, we choose N_{pi} pinning nodes according to the descending order of their degrees.

More specifically, for the network used in Fig. 1(a), we have three clusters, with 100, 300 and 600 nodes, i.e., $N_1 = 100$, $N_2 = 300$, and $N_3 = 600$, respectively. If we have total $N_p = 20$ pinning nodes, we need assign the pinning nodes to each clusters according to the cluster size, i.e., $N_{p1} = (N_1/N) \times N_p = 2$, $N_{p2} = (N_2/N) \times N_p = 6$, and $N_{p2} = (N_3/N) \times N_p = 12$. Then in each cluster i, N_{pi} nodes of largest degree are selected as pinning nodes.

To test whether our new pinning strategy can work efficiently, we apply our new pinning strategy to the same artificial network in Fig. 1(a) and redraw it in Fig. 4, where we compare the controllability of the network by adopting the RP and CP strategies when the pinning ratio is small, $f < 0.2$. It is clearly shown that the CP strategy behaves much better than the RP strategy, and also we can find the stage-like controllability of RP strategy. As the pinning ratio increases, more and more clusters will have pinning nodes, and thus the controllability of the network can be improved steadily, as we discussed in Section 3. And the question *how many nodes in the network have to be controlled* thus can be roughly discussed here. For the clustered network, especially for the clustered network with varied cluster size, the number of pinning nodes required depends on whether each cluster contains at least one pinning node. For CP strategy, pinning nodes can be covered on all clusters faster than the PP strategy, furthermore, it can be easier to control the large degree nodes than PP strategy. For

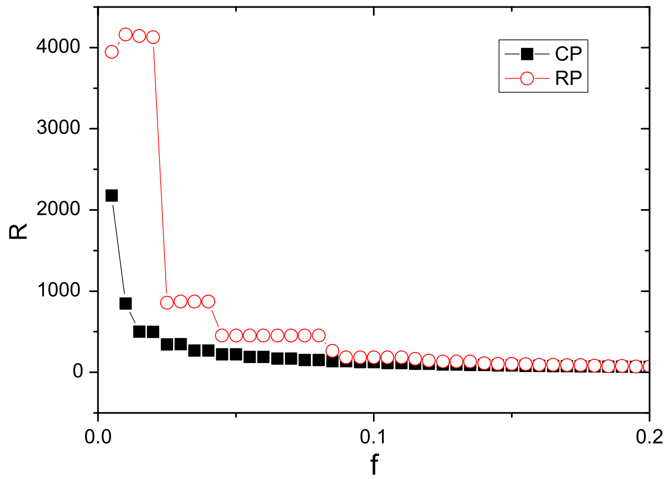

Fig. 4. (Color online) The controllability of the CP and RP strategy on the same artificial network in Fig. 1(a).

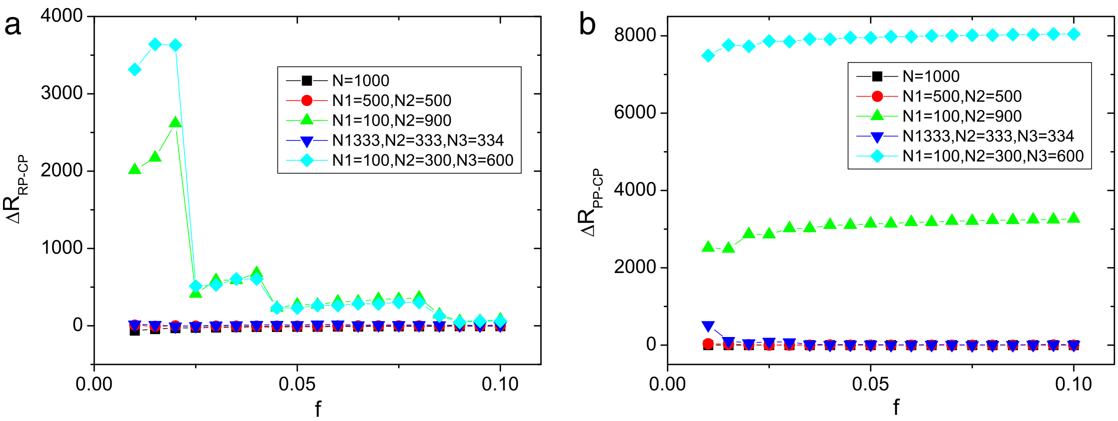

Fig. 5. (Color online) The controllability difference between different pinning strategies. (a) The controllability difference between the RP and CP strategies; (b) The controllability difference between the PP and CP strategies.

the situation of large pinning ratio, e.g., $f > 0.5$, it is found that the controllability of RP and CP are nearly the same. It is understandable because when the pinning ratio is large, most nodes with large degree are pinned, whatever by using RP or CP strategy.

In order to demonstrate the role of cluster size and compare with Ref. [13], we analyze the controllability of networks with different cluster structure. First, we compare the controllability of networks with different number of clusters, for simplification, we fix the network size as $N = 1000$. In Fig. 5, we plot the controllability difference $\Delta R = R_{stratege1} - R_{stratege2}$ as a function of pinning ratio. It is shown that, when the clusters have similar size, e.g., there are two clusters with 500 nodes in each, the controllability difference $\Delta R_{RP-CP} = R_{RP} - R_{CP}$ is close to zero. We observe the same phenomenon when we compare the PP and CP strategies. Furthermore, it is worth mentioned that the results are similar when we consider three clusters with similar size, e.g., $N_1 = 333, N_2 = 333, N_3 = 334$. However, when the clusters have quite different size, e.g., two clusters ($N_1 = 100, N_2 = 900$) and three clusters ($N_1 = 100, N_2 = 300, N_3 = 600$), the advantage of CP strategy is obvious.

Next we compare the pinning controllability with different intra cluster connection probability P_{in}, as is shown in Fig. 6. Here, the network size is fixed as $N = 1000$, and there are three clusters. To compare with Ref. [13], in Fig. 6(a), the clusters almost have the same size ($N_1 = 333, N_2 = 333, N_3 = 334$), and in Fig. 6(b), they have different size ($N_1 = 100, N_2 = 300, N_3 = 600$). As is shown in Fig. 6(a), the controllability of the network first decreases and then slowly increases as the pinning strength increases, indicating that there is an appropriate range for the pinning strength selection. These phenomena are well consist with Ref. [13]. However, the decrease–increase phenomena of the controllability become insignificant when the clusters have different size, especially for the PP strategy. In this case, the controllability is almost increases monotonically as the pinning strength increases. Meanwhile, Fig. 6(b) also shows that the CP strategy performs best, compared with the other two strategies.

For a further study, we investigate the impact of cluster size ratio on the pinning strategies, as shown in Fig. 7. Here, we divide the network ($N = 1000$) into two clusters and study the change of controllability with the size ratio, N_1/N_2, between these two clusters. Suppose N_1 is not larger than N_2, the cluster size tend to be homogeneous if the ratio goes to 1 and tends

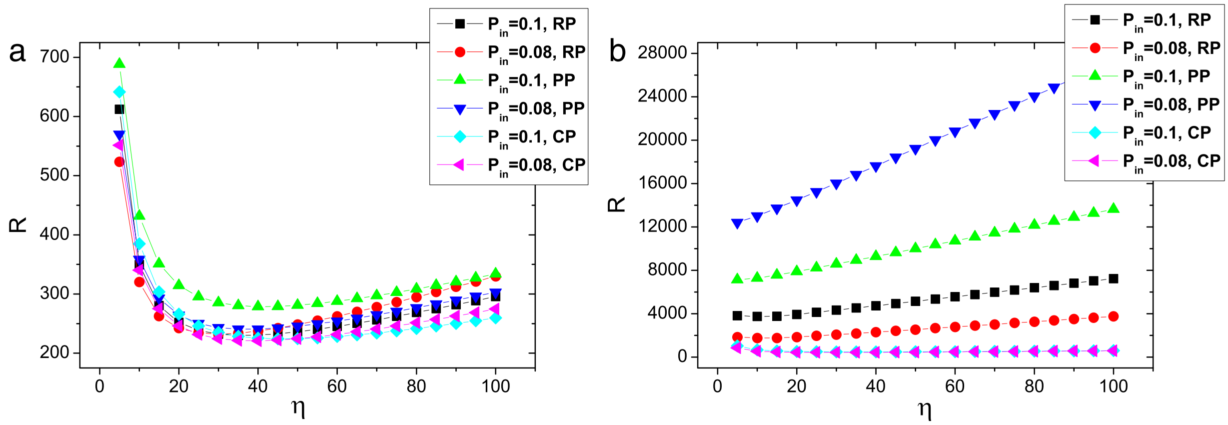

Fig. 6. The controllability of different strategies as a function of the pinning strength. The network contains three clusters (a) of similar size $N_1 = 333$, $N_2 = 333$, $N_3 = 334$, and (b) of different size $N_1 = 100$, $N_2 = 300$, $N_3 = 600$. The pinning ratio is set to 0.02.

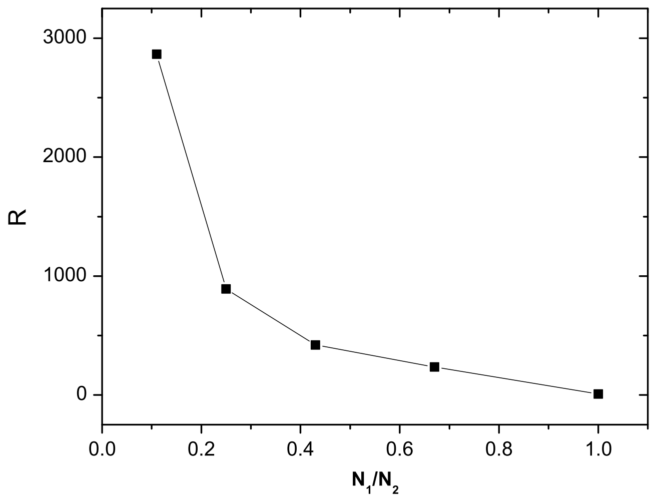

Fig. 7. The controllability difference of the RP and CP strategies as a function of the size ratio between two clusters.

to be heterogeneous if the ratio goes to 0. As is shown in Fig. 7, when the cluster size becomes more and more homogeneous, the difference between pinning strategies is getting more difficult to be distinguished, indicating that our pinning strategy can achieve better controllability only when the cluster size is heterogeneous.

So far, all our studies are based on eigenvalue and eigenvector analysis, in which many of the intrinsic properties of the node dynamics have been neglected. To justify the results of the CP strategy, we have compared the three pinning strategies on clustered network of coupled chaotic oscillators by direct simulation. The node dynamics is employed by chaotic Rössler oscillator, which described by $\mathbf{F}(\mathbf{x}) = [-y - z, x + 0.15y, z(x - 8.5) + 0.4]$. Random initial conditions are used for the oscillators and the coupling strength is set to $\varepsilon = 0.5$. Pinning control started at $t = 100$, where only a small number of nodes are pinned, $f = 0.02$, with the strength $\eta = 20$. The control performance is measured by $\delta = \sum_{i=1}^{N}[(x_i - x_T)^2 + (y_i - y_T)^2 + (z_i - z_T)^2]^{1/2}/N$, where (x_T, y_T, z_T) are the state variables of the target. As shown in Fig. 8, with the same pinning effort, the whole system can be controlled after $t = 1500$ by adopting the CP strategy, but cannot be controlled by adopting the PP and RP strategies, i.e., the control error δ stays around 3.0 and 0.04 for these two strategies, respectively. It should be noted that, in many engineering cases, the total pinning effort is finite, and if the pinning strength is fixed, the pinning effort is mainly determined by the pinning ratio. This is why we only consider a small number of pinning nodes here.

5. Discussion and conclusion

The present work is a necessary and nontrivial extension on the previous studies of pinning control on clustered network. Most of preview work had been concentrated on the clustered networks with similar cluster size, but seldom paid attention to the clustered networks with highly varied cluster size. Our findings thus will extend the knowledge of pinning control on clustered network.

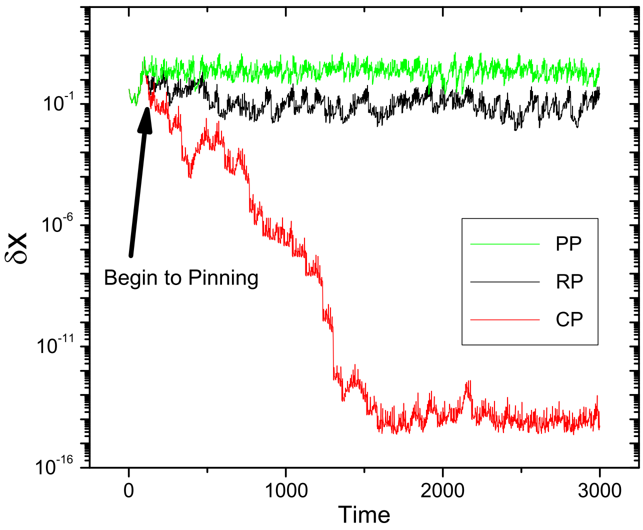

Fig. 8. (Color online) Direct simulation for the control performance of the three pinning strategies on the clustered network of coupled chaotic Rössler oscillators. The network is the same as that in Fig. 1(a).

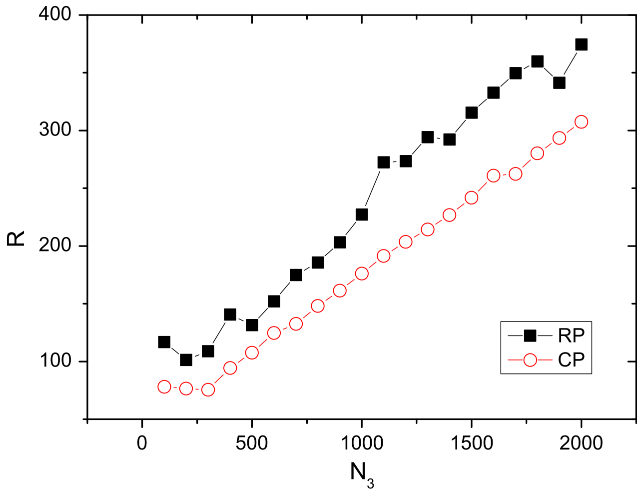

Fig. 9. (Color online) The controllability as a function of the cluster size for the CP and RP strategies. Each data is averaged over 64 realizations.

Note that the above findings can be observed when the cluster size changes. For the network used in Fig. 1(a), we now change the size of cluster three, N_3, from 100 to 2000, and compare the controllability of the CP and RP strategies. The number of pinned nodes are set as 10% of network size. In Fig. 9, it can be seen that, as the cluster size increases, the network becomes harder to be controlled, no matter which strategy is adopted. Moreover, although the cluster size weakens the network controllability, the CP strategy always behaves better than the RP strategy.

In summary, we have investigated the pinning strategy on clustered networks of varied cluster size. Our studies show that the traditional PP strategy is losing its efficiency on such networks, and even the RP strategy behave better than the PP strategy. Furthermore, by using the eigenvalue and eigenvector analysis, we find that in clustered networks, distributing the pinning nodes in all clusters, rather than in some of them, may result in a better controllability of the network. Moreover, in order to further improve the RP strategy, we have proposed a novel CP strategy, which tends to put the pinned nodes into more clusters and then choose the nodes of the largest degree in each cluster as the final pinning nodes. We find that the CP strategy preforms best when there are only a small number of pinning nodes, and this result is verified by the direct simulation on the clustered networks with coupled chaotic oscillators. Considering the universal existence of modular structure in nature, there have been many works on the control of real clustered network, e.g., Tang et al. have converted the control problem to the multiobjective problem [38] and considered the control gains [39] in a real cat brain network. However, it is our believing that the findings obtained in this work will helpful to the control of the networks with heterogeneous cluster structure in broad realistic systems, say, for instance, improve the efficiency of overall project in the Open Source Software (OSS) [40] by controlling the leaders in social network of sub-project.

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